TRANSIENT TEMPERATURE IN A POROUS TUBE

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An analytical solution is given for the transient temperature distribution in a porous tube with generalized boundary conditions at the inner and outer surfaces.

Porous cooling is an efficient method of thermal protection under conditions of high thermal loads; it is used in cooling the nozzles of rocket motors, the walls of aerodynamic tubes, experimental instruments for short-lived phenomena, etc. In connection with this, great interest attaches to the temperature distribution across the thickness of a porous wall. The steady-state problems are studied in [1-8].

It can happen in many cases that the expenditure of coolant will be excessively great if it is determined from the solution for a stationary temperature field, and will be found to be within reasonable limits if it is determined on the basis of a solution for a nonstationary system. Therefore the transient temperature of a flat porous wall is studied in [9-13].

A solution which can be used to calculate the nonstationary temperature distribution in a porous tube was obtained in [14] by the method of finite integral transformations, although its application to porous cooling was not shown. Later in [15] a solution was obtained by the method of separation of variables for the temperature of a porous tube for two different boundary conditions at the outer surface of the tube: convective heat exchange with the surrounding medium; and a constant heat flux at the surface. After uncomplicated transformations the solutions obtained [15] can be presented in a more compact form. Because the solutions in [15] are rather cumbersome and require a large amount of calculating work, in [16] it was proposed to use the method of finite integral transformations. It must be said that the solution obtained in [16] is less cumbersome than that in [15] since the authors have solved the problem with simplified boundary conditions.

An analytical solution for the nonstationary temperature distribution in a porous tube is given in the present note on the basis of a solution of the generalized transport equation given by the authors in [17].

The temperature field of a porous tube is described by the equation [15]

$$\frac{\partial\theta(\xi,\ \mathrm{Fo})}{\partial\mathrm{Fo}}=\frac{1}{\xi}\cdot\frac{\partial}{\partial\xi}\left(\xi\,\frac{\partial\theta(\xi,\ \mathrm{Fo})}{\partial\xi}\right)-\frac{2\nu}{\xi}\cdot\frac{\partial\theta(\xi,\ \mathrm{Fo})}{\partial\xi}+\mathrm{Po}(\xi),\qquad(1)$$

$$\xi_0\leqslant\xi\leqslant\xi_1,\quad \mathrm{Fo}\geqslant 0,$$

where ξ, Fo, and Po(ξ) are the dimensionless coordinate, the Fourier number, and the internal heat source, respectively, while ν is a parameter of the coolant flow rate.

Equation (1) will be solved for the boundary conditions

$$\theta(\xi,\ 0)=f_0(\xi),\qquad(2)$$

$$A_0\,\frac{\partial\theta(\xi_0,\ \mathrm{Fo})}{\partial\xi}+B_0\theta(\xi_0,\ \mathrm{Fo})=b_0,\qquad(3)$$

$$A_1\,\frac{\partial\theta(\xi_1,\ \mathrm{Fo})}{\partial\xi}+B_1\theta(\xi_1,\ \mathrm{Fo})=b_1,\qquad(4)$$

where A_0, B_0, A_1, B_1, b_0, and b_1 are fixed constants. By an appropriate choice of these constants, it is possible, without difficulty, to obtain as particular cases the solutions given in [14-16], where $f_0(\xi)$ = const and Po(ξ) = 0.

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We note in passing that an approximate calculation of the three-dimensional heat-conduction problem is reduced to an analogous one-dimensional problem in [18].

Equation (1) can be presented in the form

$$\xi^{1-2\nu} \frac{\partial\theta(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial}{\partial\xi}\left(\xi^{1-2\nu} \frac{\partial\theta(\xi, \mathrm{Fo})}{\partial\xi}\right) + \xi^{1-2\nu}\,\mathrm{Po}(\xi), \tag{5}$$

from which it is seen that it is a particular case of a more general transport equation the solution of which is given in [17] in the form of series of eigenfunctions.

In the case under examination the eigenvalues μ_i and the eigenfunctions $\psi_i(\xi)$ are determined by the following Sturm–Liuville problem:

$$\psi''(\xi) + \frac{1-2\nu}{\xi}\psi'(\xi) + \mu^2\psi(\xi) = 0, \tag{6}$$

$$A_0\psi'(\xi_0) + B_0\psi(\xi_0) = 0, \tag{7}$$

$$A_1\psi'(\xi_1) + B_1\psi(\xi_1) = 0. \tag{8}$$

Comparing (6) with the generalized Bessel equation obtained by Douglas and presented in [19], we find

$$\psi(\xi) = \xi^\nu\{C_1 J_\nu(\mu\xi) + C_2 Y_\nu(\mu\xi)\}. \tag{9}$$

The boundary condition (7) is satisfied if we set

$$C_1 = B_0 Y_\nu(\mu\xi_0) + \mu A_0 Y_{\nu-1}(\mu\xi_0), \tag{10}$$

$$C_2 = -B_0 J_\nu(\mu\xi_0) - \mu A_0 J_{\nu-1}(\mu\xi_0), \tag{11}$$

while (8) gives the following characteristic equation for determining μ_i:

$$\frac{B_0 Y_\nu(\mu\xi_0) + \mu A_0 Y_{\nu-1}(\mu\xi_0)}{B_0 J_\nu(\mu\xi_0) + \mu A_0 J_{\nu-1}(\mu\xi_0)} = \frac{B_1 Y_\nu(\mu\xi_1) + \mu A_1 Y_{\nu-1}(\mu\xi_1)}{B_1 J_\nu(\mu\xi_1) + \mu A_1 J_{\nu-1}(\mu\xi_1)}. \tag{12}$$

Then the solution of [17] is obtained in the following form:

$$\theta(\xi, \mathrm{Fo}) = \Bigg\{ b_0 A_1 \xi_0^{1-2\nu} - b_1 A_0 \xi_1^{1-2\nu} + (\xi_0\xi_1)^{1-2\nu}\left[b_0 B_1 \int_{\xi_0}^{\xi_1} \frac{d\xi}{\xi^{1-2\nu}} + (b_1 B_0 - b_0 B_1)\int_{\xi_0}^{\xi}\frac{d\xi}{\xi^{1-2\nu}}\right] + \left[B_0\xi_0^{1-2\nu}\int_{\xi_0}^{\xi}\frac{d\xi}{\xi^{1-2\nu}} - A_0\right]$$

$$\times\left[A_1\int_{\xi_0}^{\xi_1}\xi^{1-2\nu}\,\mathrm{Po}(\xi)\,d\xi + B_1\xi_1^{1-2\nu}\int_{\xi_0}^{\xi_1}\frac{1}{\xi^{1-2\nu}}\left(\int_{\xi_0}^{\xi}\xi^{1-2\nu}\,\mathrm{Po}(\xi)\,d\xi\right)d\xi\right]\Bigg\}$$

$$\times\left\{A_1 B_0 \xi_0^{1-2\nu} - A_0 B_1 \xi_1^{1-2\nu} + B_0 B_1 (\xi_0\xi_1)^{1-2\nu}\int_{\xi_0}^{\xi_1}\frac{d\xi}{\xi^{1-2\nu}}\right\}^{-1}$$

$$-\int_{\xi_0}^{\xi}\frac{1}{\xi^{1-2\nu}}\left(\int_{\xi_0}^{\xi}\xi^{1-2\nu}\,\mathrm{Po}(\xi)\,d\xi\right)d\xi + \sum_{i=1}^{\infty} G_i\psi_i(\xi)\, e^{-\mu_i^2 \mathrm{Fo}}$$

$$\times\left\{\int_{\xi_0}^{\xi_1}\xi^{1-2\nu}\psi_i(\xi) f_0(\xi)\,d\xi - \frac{1}{\mu_i^2}\left[b_1\frac{2}{\pi\xi_1^\nu}\cdot\frac{B_0 J_\nu(\mu_i\xi_0) + \mu_i A_0 J_{\nu-1}(\mu_i\xi_0)}{B_1 J_\nu(\mu_i\xi_1) + \mu_i A_1 J_{\nu-1}(\mu_i\xi_1)} - b_0\frac{2}{\pi\xi_0^\nu} + \int_{\xi_0}^{\xi_1}\xi^{1-2\nu}\psi_i(\xi)\,\mathrm{Po}(\xi)\,d\xi\right]\right\}, \tag{13}$$

where

$$G_i = \frac{\pi^2\mu_i^2}{2}\left\{\left[\frac{B_0 J_\nu(\mu_i\xi_0) + \mu_i A_0 J_{\nu-1}(\mu_i\xi_0)}{B_1 J_\nu(\mu_i\xi_1) + \mu_i A_1 J_{\nu-1}(\mu_i\xi_1)}\right]^2\left[\mu_i^2 A_1^2 + B_1\left(B_1 + A_1\frac{2\nu}{\xi_1}\right)\right] - \left[\mu_i^2 A_0^2 + B_0\left(B_0 + A_0\frac{2\nu}{\xi_0}\right)\right]\right\}^{-1}. \tag{14}$$

For boundary conditions of the second type at both surfaces of the tube, i.e., when $b_0 = b_1 = 0$, the solution of [17] has the form

$$\begin{aligned}
\theta(\xi,\mathrm{Fo}) = {} & \frac{1}{\int_{\xi_0}^{\xi_1}\xi^{1-2\nu}d\xi}\Bigg\{\int_{\xi_0}^{\xi_1}\xi^{1-2\nu}f_0(\xi)\,d\xi + \Bigg[\frac{b_1}{A_1}\xi_1^{1-2\nu} - \frac{b_0}{A_0}\xi_0^{1-2\nu} \\
& + \int_{\xi_0}^{\xi_1}\xi^{1-2\nu}\mathrm{Po}(\xi)\,d\xi\Bigg]\Bigg[\mathrm{Fo} + \int_{\xi_0}^{\xi}\frac{1}{\xi^{1-2\nu}}\left(\int_{\xi_0}^{\xi}\xi^{1-2\nu}d\xi\right)d\xi \\
& - \frac{\int_{\xi_0}^{\xi_1}\xi^{1-2\nu}\left(\int_{\xi_0}^{\xi}\frac{1}{\xi^{1-2\nu}}\left[\int_{\xi_0}^{\xi}\xi^{1-2\nu}d\xi\right]d\xi\right)d\xi}{\int_{\xi_0}^{\xi_1}\xi^{1-2\nu}d\xi}\Bigg] - \frac{b_0}{A_0}\xi_0^{1-2\nu}\int_{\xi_0}^{\xi_1}\xi^{1-2\nu} \\
& \times\left(\int_{\xi_0}^{\xi}\frac{d\xi}{\xi^{1-2\nu}}\right)d\xi + \int_{\xi_0}^{\xi_1}\xi^{1-2\nu}\left(\int_{\xi_0}^{\xi}\frac{1}{\xi^{1-2\nu}}\left[\int_{\xi_0}^{\xi}\xi^{1-2\nu}\mathrm{Po}(\xi)\,d\xi\right]d\xi\right)d\xi\Bigg\} \\
& + \frac{b_0}{A_0}\xi_0^{1-2\nu}\int_{\xi_0}^{\xi}\frac{d\xi}{\xi^{1-2\nu}} - \int_{\xi_0}^{\xi_1}\frac{1}{\xi^{1-2\nu}}\left(\int_{\xi_0}^{\xi}\xi^{1-2\nu}\mathrm{Po}(\xi)\,d\xi\right)d\xi \\
& + \sum_{i=1}^{\infty}G_i\psi_i(\xi)e^{-\mu_i^2\mathrm{Fo}}\Bigg\{\int_{\xi_0}^{\xi_1}\xi^{1-2\nu}\psi_i(\xi)f_0(\xi)\,d\xi - \frac{1}{\mu_i^2}\Bigg[b_1\frac{2}{\pi\xi_1^{\nu}} \\
& \times\frac{A_0J_{\nu-1}(\mu_i\xi_0)}{A_1J_{\nu-1}(\mu_i\xi_1)} - b_0\frac{2}{\pi\xi_0^{\nu}} + \int_{\xi_0}^{\xi_1}\xi^{1-2\nu}\psi_i(\xi)\mathrm{Po}(\xi)\,d\xi\Bigg]\Bigg\}.
\end{aligned}\tag{15}$$

From Eqs. (13) and (15) for $\nu = 1/2$, 0, and $-1/2$ solutions may also be obtained for bodies of simple form without taking into account convective heat transport.

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